

FMLtoHOL (version 1.0): Automating First-order Modal Logics with LEO-II and Friends^{*}

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Abstract. A converter from first-order modal logics to classical higher-order logic is presented. This tool enables the application of off-the-shelf higher-order theorem provers and model finders for reasoning within first-order modal logics. The tool supports logics K, K4, D, D4, T, S4, and S5 with respect to constant, varying and cumulative domain semantics.

1 Introduction

First-order modal logics (FMLs) [4] have many applications, e.g., in planning, natural language processing, program verification, querying knowledge bases, and modeling communication. These applications motivate the use of *automated theorem proving* (ATP) systems for FMLs. Several new FML ATP systems, including two FMLtoHOL-based solutions, have recently been provided [1].

This paper describes the FMLtoHOL tool, which automatically converts problems in FML, formulated in the new **qmf**-syntax [5] (which extends the TPTP **fol**-syntax [7] with operators **#box** and **#dia**), into problems in classical higher-order logic (HOL), formulated in **thf0**-syntax [6]. FMLtoHOL exploits and implements a semantic embedding of constant domain FML in HOL [2,3]. Moreover, the tool extends this embedding to varying and cumulative domain semantics.

FMLtoHOL thus turns any **thf0**-compliant HOL ATP system — such as LEO-II³ and Satallax³ — into a flexible ATP system for FML. At present FMLtoHOL supports modal logics $L := \{K, K4, D, D4, T, S4, S5\}$, all with respect to constant, varying and cumulative domain semantics. Extending the tool to other normal FMLs and their combinations is straightforward.

In the remainder the language of FML is fixed as: $F, G ::= P(t_1, \dots, t_n) \mid \neg F \mid F \wedge G \mid F \vee G \mid F \Rightarrow G \mid \Box F \mid \Diamond F \mid \forall x F \mid \exists x F$. The symbols P are n -ary ($n \geq 0$) relation constants which are applied to terms t_1, \dots, t_n . The t_i ($0 \leq i \leq n$) are ordinary first-order terms and they may contain function symbols. The formula $(\forall x \Box f x) \Rightarrow \Box \forall x f x$ is used as an example, it is referred to as **E1**. In constant domain (resp. varying domain) semantics **E1** is a theorem (resp. countersatisfiable) for logics L . In cumulative domain semantics **E1** is a theorem for S5 and countersatisfiable for the other logics in L .

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³ Cf. www.leoprover.org and www.ps.uni-saarland.de/~cebrown/satallax/

2 Theory of FMLtoHOL

FMLtoHOL exploits the fact that Kripke structures can be elegantly embedded in HOL [2,3]: FML propositions F are associated with HOL terms F_ρ of predicate type $\rho := \iota \rightarrow o$. Type o denotes the set of truth values and type ι is associated with the domain of possible worlds. Thus, the application $(F_\rho w_\iota)$ corresponds to the evaluation of FML proposition F in world w . Consequently, validity is formalized as $vld_{\rho \rightarrow o} = \lambda F_\rho \forall w_\iota F w$. Classical connectives like \neg and \vee are simply lifted to type ρ as follows: $\neg_{\rho \rightarrow \rho} = \lambda F_\rho \lambda w_\iota \neg F w$ and $\vee_{\rho \rightarrow \rho \rightarrow \rho} = \lambda F_\rho \lambda G_\rho \lambda w_\iota (F w \vee G w)$. \Box is modeled as $\Box_{\rho \rightarrow \rho} = \lambda F_\rho \lambda w_\iota \forall v_\iota (\neg R w v \vee F v)$, where constant symbol $R_{\iota \rightarrow \rho}$ denotes the accessibility relation of the \Box operator, which remains unconstrained in logic K. Further logical connectives are defined as usual: $\wedge = \lambda F_\rho \lambda G_\rho \neg(\neg F \vee \neg G)$, $\Rightarrow = \lambda F_\rho \lambda G_\rho (\neg F \vee G)$, $\Diamond = \lambda F_\rho \neg \Box \neg F$. To obtain e.g. modal logics D, T, S4, and S5, R is axiomatized as serial, reflexive, reflexive and transitive, and an equivalence relation, respectively. Arbitrary normal modal logics extending K can be axiomatized this way.

For individuals a further base type μ is reserved in HOL. Universal quantification $\forall x F$ is introduced as syntactic sugar for $\Pi \lambda x F$, where Π is defined as follows: $\Pi_{(\mu \rightarrow \rho) \rightarrow \rho} = \lambda H_{\mu \rightarrow \rho} \lambda w_\iota \forall x_\mu H x w$. For existential quantification, $\Sigma = \lambda H_{\mu \rightarrow \rho} \neg \Pi \lambda x_\iota \neg H x$ is introduced. $\exists x F$ is then syntactic sugar for $\Sigma \lambda x F$. n -ary relation symbols P , n -ary function symbols f and individual constants c in FML obtain types $\mu_1 \rightarrow \dots \rightarrow \mu_n \rightarrow \rho$, $\mu_1 \rightarrow \dots \rightarrow \mu_n \rightarrow \mu_{n+1}$ (both with $\mu_i = \mu$ for $0 \leq i \leq n+1$) and μ , respectively.

For any FML formula F holds: F is a valid in modal logic K for constant domain semantics if and only if $vld F_\rho$ is valid in HOL for Henkin semantics. This correspondence provides the foundation for proof automation of FMLs with HOL-ATP systems. The correspondence follows from [2], where a more general result is shown for FMLs with additional quantification over Boolean variables.

The above approach is adopted for varying domain semantics as follows: 1. Π is now defined as $\Pi = \lambda H_{\mu \rightarrow \rho} \lambda w_\iota \forall x_\mu \text{exIn} w x w \Rightarrow H x w$, where relation $\text{exIn} w_{\mu \rightarrow \iota \rightarrow o}$ (for ‘exists in world’) relates individuals with worlds. 2. The non-emptiness axiom $\forall w_\iota \exists x_\mu \text{exIn} w x w$ for these individual domains is added. 3. For each individual constant symbol c an axiom $\forall w_\iota \text{exIn} w c w$ is postulated; these axioms enforce the designation of c in the individual domain of each world w . Analogous designation axioms are required for function symbols.

For cumulative domain semantics the axiom $\forall x_\mu \forall v_\iota \forall w_\iota \text{exIn} w x v \wedge R v w \Rightarrow \text{exIn} w x w$ is additionally postulated. It states that the individual domains are increasing along relation R .

3 Implementation and Functionality of FMLtoHOL

FMLtoHOL is implemented as part of the TPTP2X tool [7]. TPTP2X is a multi-functional utility for generating, transforming, and reformatting TPTP problem files. It is written in Prolog and it can be easily modified and extended.

The tool is invoked as “`tptp2X -f thf:<logic>:<domain> <qmf-file>`” where $\text{<logic>} \in \{K, K4, D, D4, T, S4, S5\}$ and $\text{<domain>} \in \{\text{const, vary, cumul}\}$.

To illustrate its use it is assumed that file `E1.qmf` contains `E1` in `qmf`-syntax:

```
qmf(con,conjecture,(
  ( ! [X] : ( #box : ( f(X) ) ) ) => ( #box : ( ! [X] : ( f(X) ) ) ) ) ).
```

“tptp2X -f thf:d:const E1.qmf” generates the corresponding HOL problem file E1.thf in thf-syntax⁴ [6] for constant domain logic D:

```
%----Include axioms for modal logic D under constant domains
include('Axioms/LCL013^0.ax.const').
include('Axioms/LCL013^2.ax').
%-----
thf(f_type,type,( f: mu > $i > $o )). % type declaration for constant f

thf(prove,conjecture,( mvalid @
  ( mimplies @ ( mforall_ind @ ^ [X: mu] : ( mbox_d @ ( f @ X ) ) )
    @ ( mbox_d @ ( mforall_ind @ ^ [X: mu] : ( f @ X ) ) ) ) ) ).
```

The included axiom files contain the definitions of the logical connectives as outlined in Sect. 2. For example, the definition for `mforall_ind` (which realizes Π for constant domain semantics) is given in `LCL013^0.ax.const`:

```
thf(mforall_ind,definition,( mforall_ind =
  ( ^ [Phi: mu > $i > $o, W: $i] : ! [X: mu] : ( Phi @ X @ W ) ) ) ).
```

`LCL013^2.ax` contains the definition of the serial \Box operator in logic D:

```
thf(mbox_d,definition,( mbox_d =
  ( ^ [Phi: $i > $o, W: $i] :
    ! [V: $i] : ( ~ ( rel_d @ W @ V ) | ( Phi @ V ) ) ) ) ).
```

```
thf(a1,axiom,( mserial @ rel_d ) ).
```

Similar definitions are provided in the included axiom files for the other logical connectives and for auxiliary terms like `mserial`. The HOL ATP systems LEO-II and Satallax when applied to E1.thf find a proof within a few milliseconds.

When `FMLtoHOL` is called with option “-f thf:s5:vary” a modified file E1.thf is created containing a conjecture identical to above except that `mbox_d` is replaced by `mbox_s5`. Moreover, E1.thf now includes different axiom files `LCL013^0.ax.vary` and `LCL013^6.ax`. The former contains a modified definition of `mforall_ind`, which incorporates an explicit ‘exists in world’ condition:

```
thf(mforall_ind,definition,( mforall_ind =
  ( ^ [Phi: mu > $i > $o, W: $i] :
    ! [X: mu] : ( ( exists_in_world @ X @ W ) => ( Phi @ X @ W ) ) ) ) ).
```

```
thf(nonempty_ax,axiom,(
  ! [V : $i] : ? [X : mu] : (exists_in_world @ X @ V))) .
```

The latter axiom specifies the domains of existing objects as non-empty for all worlds V . Axiom file `LCL013^6.ax` specifies `mbox_s5` as follows:

⁴ Some explanations: \wedge is λ -abstraction and $@$ an (explicit) application operator. $!$, $?$, \sim , $|$, and \Rightarrow encode universal and existential quantification, negation, disjunction and implication in HOL. $\mu > \$i > \o encodes the HOL type $\mu \rightarrow \iota \rightarrow o$. `mimplies`, `mforall_ind`, and `mbox_d` are embedded logical connectives as described in Sect. 2. Their denotation is fixed by adding definition axioms; see e.g. `mforall_ind` below.

```

thf(mbox_s5,definition,( mbox_s5 =
  ( ~ [Phi: $i > $o,W: $i] :
    ! [V: $i] : ( ~ ( rel_s5 @ W @ V ) | ( Phi @ V ) ) ) ).

thf(a1,axiom,( mreflexive @ rel_s5 )).
thf(a2,axiom,( mtransitive @ rel_s5 )).
thf(a3,axiom,( msymmetric @ rel_s5 )).

```

For the modified problem Satallax finds a counter model within milliseconds.

4 Discussion and Outlook

The FMLtoHOL has been applied and evaluated in combination with the HOL ATP systems Satallax and LEO-II; cf. [1] for details. In this case study the approach has also been compared with other, heterogeneous FML ATP systems. The FMLtoHOL based solution has the best coverage (and it can easily be extended to other modal logics and their combinations) and it is second best in overall performance behind the clausal connection prover MleanCoP⁵.

Future work includes several optimizations of the tool, extensions for multi-modal logics (which it already partly supports), and further case studies. These case studies should evaluate the tool also in combination with other thf0-compliant HOL provers and model finders as outlined in [6]: TPS, Isabelle, Refute and Nitpick.

A recent observation is that the HOL model finders Satallax, Refute and Nitpick apparently work well for constant and varying domain semantics but have problems to find counter models for cumulative domain semantics.

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References

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⁵ Cf. the information at <http://www.iltp.de/qmltp/systems.html>